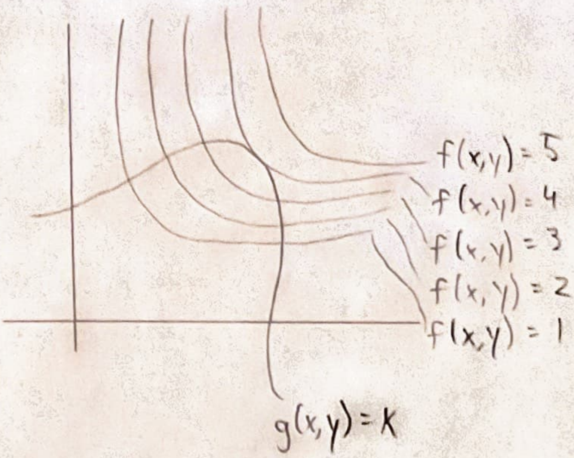
Lecture 17

Lagrange Multipliers



The function $f(x,y)$ is maximized on the constraint $g(x,y)=K$ when they "just touch". That is, they have a common tangent. Mathematically this means that an extreme point, (x_0, y_0):

$$\nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0).$$

The scalar, λ, is called a lagrange multiplier.

This can be extended to functions of three variables:

$$\nabla f(x_0, y_0, z_0) = \lambda \nabla g(x_0, y_0, z_0).$$

The Method of Lagrange Multipliers

To find an extreme value of an objective function, f, with a given constraint, $g = K$:

1. Assume f has an extreme value on $g = K$.
2. Solve the equations:

$$g(x,y) = K$$

$$\nabla f(x,y) = \lambda \nabla g(x,y) \left\{ \begin{array}{l} f_x(x,y) = \lambda g_x(x,y) \\ f_y(x,y) = \lambda g_y(x,y) \end{array} \right.$$

3. Find the value of $f(x,y)$ at all points you find in step 2. The max(min) will be the max(min) of these values.

Ex. 1 Find the maximum value of a rectangular box, without a lid, made out of $12\ m^2$ of cardboard.

$$f(x,y,z) = xyz$$

$$g(x,y,z) = xy + 2yz + 2xz = 12$$

$f_x = yz$, $f_y = xz$, $f_z = xy$

$g_x = y + 2z$, $g_y = 2z + x$, $g_z = 2(x+y)$

1: $yz = \lambda(y + 2z)$

2: $xz = \lambda(2z + x)$ note $\lambda \neq \emptyset$ or g would not hold

3: $xy = \lambda(2x + 2y)$

Multiply by $x, y, + z$ respectively:

4) $xyz = \lambda x(y + 2z)$

5) $xyz = \lambda y(2z + x)$

6) $xyz = \lambda z(2x + 2y)$

Equating 4) + 5)

$$\cancel{\lambda} x(y + 2z) = \cancel{\lambda} y(2z + x)$$
$$\cancel{xy} + 2xz = 2yz + \cancel{xy}$$
$$x\cancel{z} = y\cancel{z} \qquad z \neq \emptyset$$
$$x = y$$

Equating 5) + 6)

$$y(2z + x) = z(2x + 2y)$$
$$2yz + xy = 2xz + \cancel{2yz}$$
$$y = 2z$$

$x = y = 2z$

Plug into $g(x,y)$

$$2xz + 2yz + xy = 12$$
$$4z^2 + 4z^2 + 4z^2 = 12$$
$$12z^2 = 12$$
$$z^2 = 1$$
$$z = \pm 1$$

z must be positive. So,

$z = 1$

$x = y = 2z$

$x = 2$

$y = 2$

$z = 1$

$f(2,2,1) = 4m^3$ is the max volume.

Ex. 2 Find the extreme values of $f(x,y) = x^2 + 2y^2$ on the disk $x^2 + y^2 \leq 1$

We have to check the interior first. So we find the critical points.

$f_x = 2x = \emptyset$ $\quad$ $f_y = 4y = \emptyset$

$x = \emptyset$ $\quad$ $y = \emptyset$

$f(\emptyset, \emptyset) = \emptyset$

Now the boundary

$g(x,y) = 1$

$g_x = 2x$ $\quad$ $g_y = 2y$

1) $2x = \lambda 2x$

$\lambda = 1$ or $x = \emptyset$

2) $4y = \lambda 2y$

$2y = \lambda y$

if $x = \emptyset \rightarrow g(\emptyset, y) = y^2 = 1 \rightarrow y = \pm 1$ $\quad (\emptyset, 1)$ & $(\emptyset, -1)$

if $\lambda = 1 \rightarrow y = \emptyset \rightarrow g(x, \emptyset) = x^2 = 1 \rightarrow x = \pm 1$ $\quad (1, \emptyset)$ & $(-1, \emptyset)$

$f(\emptyset, 1) = 2$ $\quad f(\emptyset, -1) = 2$ $\quad f(1, \emptyset) = 1$ $\quad f(-1, \emptyset) = 1$

max @ $f(\emptyset, \pm 1) = 2$ $\quad$ min @ $f(\emptyset, \emptyset) = \emptyset$

Two Constraints

If we wish to optimize $f(x,y,z)$ given two constraints, $g(x,y,z)=K$ & $h(x,y,z)=c$, Lagrange's method becomes solving:

$$\nabla f(x,y,z)=\lambda \nabla g(x,y,z)+\mu \nabla h(x,y,z)$$
$$g(x,y,z)=K$$
$$h(x,y,z)=c$$

Or,

$$f_x=\lambda g_x+\mu h_x$$
$$f_y=\lambda g_y+\mu h_y$$
$$f_z=\lambda g_z+\mu h_z$$
$$g(x,y,z)=K$$
$$h(x,y,z)=c$$